On the intermittency front of stochastic heat equation driven by colored noises

Yaozhong Hu[*], Jingyu Huang and David Nualart[†]
Department of Mathematics
University of Kansas
Lawrence, Kansas, 66045 USA

Abstract

We study the propagation of high peaks (intermittency front) of the solution to a stochastic heat equation driven by multiplicative centered Gaussian noise in $\mathbb{R}^d$. The noise is assumed to have a general homogeneous covariance in both time and space, and the solution is interpreted in the senses of the Wick product. We give some estimates for the upper and lower bounds of the propagation speed, based on a moment formula of the solution. When the space covariance is given by a Riesz kernel, we give more precise bounds for the propagation speed.

1 Introduction

We consider the stochastic heat equation in $\mathbb{R}^d$ driven by a general multiplicative centered Gaussian noise (parabolic Anderson model)

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + \lambda u \diamond \dot{W}\,, \tag{1.1}$$

with a continuous and nonnegative initial condition u_0 of compact support. The covariance of the noise $\dot{W}$ can be informally written as

$$\mathbb{E}\left[\dot{W}_{t,x}\dot{W}_{s,y}\right] = \gamma(s-t)\Lambda(x-y)\,,$$

and the product appearing in (1.1) is interpreted in the Wick sense.

In this paper we are interested in the position of the high peaks that are farthest away from the origin. The propagation of the farthest high peaks was first considered by Conus and Khoshnevisan in [4] for a one dimensional heat equation driven by space-time white noise, where it is shown that there are intermittency fronts that move linearly with time as αt. Namely, for any fixed $p \in [2,\infty)$, if α is sufficiently small, then the quantity $\sup_{|x|>\alpha t}\mathbb{E}(|u(t,x)|^p)$ grows exponentially fast as t tends to ∞; whereas the preceding quantity vanishes exponentially fast if α is sufficiently large. To be more precise, the authors of [4] define for every $\alpha > 0$,

$$\mathscr{S}(\alpha) := \limsup_{t\to\infty} \frac{1}{t} \sup_{|x|>\alpha t} \log \mathbb{E}(|u(t,x)|^p)\,, \tag{1.2}$$

and think of α_L as an intermittency lower front if $\mathscr{S}(\alpha) < 0$ for all $\alpha > \alpha_L$, and of α_U as an intermittency upper front if $\mathscr{S}(\alpha) > 0$ whenever $\alpha < \alpha_U$. In [4] it is shown that for each real

[*]Y. Hu is partially supported by a grant from the Simons Foundation #209206.
[†]D. Nualart is supported by the NSF grant DMS1208625 and the ARO grant FED0070445.
Keywords: Stochastic heat equation, Feynman-Kac formula, Intermittency front, Malliavin calculus, comparison principle.

number $p \geq 2$, $0 < \alpha_U \leq \alpha_L < \infty$, and when $p = 2$, some bounds for α_L and α_U are given. In a later work by Chen and Dalang [1], it is proved that when $p = 2$, there exists a critical number $\alpha^* = \frac{\lambda^2}{2}$ such that $\mathscr{S}(\alpha) < 0$ when $\alpha > \alpha^*$ while $\mathscr{S}(\alpha) > 0$ when $\alpha < \alpha^*$ (this property was first conjectured in [4]). See also [9] for a discussion of these facts.

This paper is inspired by the aforementioned works. We are interested in the multidimensional stochastic heat equation driven by a colored noise, both in space and time, when the solution is interpreted in the Wick sense. Our analysis will be based on the pth moment formula and Wiener chaos expansion of the solution, obtained in [7], as well as some small ball estimates. Due to the presence of the time covariance, the propagation speed of the farthest high peaks may not be linear. Thus, in contrast to (1.2), the inequality $|x| > \alpha t$ there needs to be replaced by $|x| > \alpha t \theta_t$ for some suitable function θ_t (see Theorems 3.1 and 3.4 below for precise choice of θ_t). When Λ is the Riesz kernel, a better estimate of the intermittency lower front is obtained in Proposition 3.9. We would like to mention the work [2], where another nonlinear propagation speed (growth indices of exponential type) is studied.

This paper is organized as follows: In Section 2, we set up some preliminaries for the structure of our Gaussian noises in equation (1.1) and present some elements of Malliavin calculus. We also prove the non-negativity of the solution to equation (1.1). Section 3 contains the main results of this paper, where we obtain some upper and lower bounds for the growth index. In the special case when the space covariance is a Riesz kernel, we give a more detailed computation for the upper bound of the growth index, and we see that the orders of λ and p in the estimate of the growth index are sharp.

2 Preliminaries

We first introduce some basic notions. The Fourier transform is defined with the normalization

$$\mathcal{F}u(\xi) = \int_{\mathbb{R}^d} e^{-\imath\langle \xi, x\rangle} u(x)dx,$$

so that the inverse Fourier transform is given by $\mathcal{F}^{-1}u(\xi) = (2\pi)^{-d}\mathcal{F}u(-\xi)$. We denote by $\mathcal{D}((0,\infty)\times\mathbb{R}^d)$ the space of infinitely differentiable functions with compact support on $(0,\infty)\times\mathbb{R}^d$.

On a complete probability space $(\Omega, \mathcal{F}, P)$ we consider a Gaussian noise W encoded by a centered Gaussian family $\{W(\varphi),\, \varphi \in \mathcal{D}((0,\infty)\times\mathbb{R}^d)\}$, whose covariance structure is given by

$$\mathbb{E}\left[W(\varphi)\, W(\psi)\right] = \int_{\mathbb{R}_+^2\times\mathbb{R}^{2d}} \varphi(s,x)\psi(t,y)\gamma(s-t)\Lambda(x-y)dxdydsdt, \tag{2.1}$$

where $\gamma : \mathbb{R} \to \mathbb{R}_+$ and $\Lambda : \mathbb{R}^d \to \mathbb{R}_+$ are non-negative definite functions. We also assume that the Fourier transform $\mathcal{F}\Lambda = \mu$ is a tempered measure, that is, there is an integer $m \geq 1$ such that $\int_{\mathbb{R}^d}(1+|\xi|^2)^{-m}\mu(d\xi) < \infty$. Our results also cover the case where γ (or Λ if $d = 1$) is the Dirac delta function, which corresponds to the time (or space) white noise.

Let $\mathcal{H}$ be the completion of $\mathcal{D}((0,\infty)\times\mathbb{R}^d)$ endowed with the inner product

$$\begin{aligned}\langle \varphi, \psi\rangle_{\mathcal{H}} &= \int_{\mathbb{R}_+^2\times\mathbb{R}^{2d}} \varphi(s,x)\psi(t,y)\gamma(s-t)\Lambda(x-y)\, dxdydsdt \\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}_+^2\times\mathbb{R}^d} \mathcal{F}\varphi(s,\xi)\overline{\mathcal{F}\psi(t,\xi)}\gamma(s-t)\mu(d\xi)\, dsdt,\end{aligned} \tag{2.2}$$

where $\mathcal{F}\varphi$ refers to the Fourier transform with respect to the space variable only. The mapping $\varphi \to W(\varphi)$ defined in $\mathcal{D}((0,\infty)\times\mathbb{R}^d)$ extends to a linear isometry between $\mathcal{H}$ and the Gaussian

space spanned by W. We will denote this isometry by

$$W(\phi) = \int_0^\infty \int_{\mathbb{R}^d} \phi(t,x) W(dt,dx)$$

for $\phi \in \mathcal{H}$. Notice that if ϕ and ψ are in $\mathcal{H}$, then $\mathbb{E}\left[W(\phi)W(\psi)\right] = \langle \phi, \psi \rangle_{\mathcal{H}}$.

We shall make a standard assumption on the spectral measure μ, which will prevail until the end of the paper.

Hypothesis 2.1 *The measure μ satisfies the following integrability condition:*

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty. \tag{2.3}$$

Now we state some basic facts about Malliavin calculus. For a detailed account on this theory, we refer to [11]. We will denote by D the Malliavin derivative. That is, if F is a smooth and cylindrical random variable of the form

$$F = f(W(\phi_1), \dots, W(\phi_n)),$$

with $\phi_i \in \mathcal{H}$, $f \in C_p^\infty(\mathbb{R}^n)$ (namely, f and all its partial derivatives have polynomial growth), then DF is the $\mathcal{H}$-valued random variable defined by

$$DF = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(W(\phi_1), \dots, W(\phi_n))\phi_j .$$

The operator D is closable from $L^p(\Omega)$ into $L^p(\Omega; \mathcal{H})$ for any $p \geq 1$ and we define the Sobolev space $\mathbb{D}^{1,p}$ as the closure of the space of smooth and cylindrical random variables under the norm

$$\|F\|_{1,p} = \left(\mathbb{E}[|F|^p] + \mathbb{E}[\|DF\|_{\mathcal{H}}^p]\right)^{\frac{1}{p}} .$$

We denote by δ the adjoint of the derivative operator given by the duality formula

$$\mathbb{E}\left[\delta(u)F\right] = \mathbb{E}\left[\langle DF, u\rangle_{\mathcal{H}}\right], \tag{2.4}$$

for any $F \in \mathbb{D}^{1,2}$ and any element $u \in L^2(\Omega; \mathcal{H})$ in the domain of δ. The operator δ is also called the *Skorohod integral* because in the case of the Brownian motion, it coincides with an extension of the Itô integral introduced by Skorohod. We will make use of the notation

$$\delta(u) = \int_0^t \int_{\mathbb{R}^d} u(s,y)\delta W_{s,y}.$$

If $F \in \mathbb{D}^{1,2}$ and ϕ is an element of $\mathcal{H}$, then $F\phi$ is Skorohod integrable and, by definition, the Wick product equals the Skorohod integral of Fh, that is,

$$\delta(F\phi) = F \diamond W(\phi). \tag{2.5}$$

In view of this definition, the mild solution to equation (1.1) will be formulated below in terms of the Skorohod integral.

Next we give a short account of Wiener chaos expansion. For any integer $n \geq 0$ we denote by $\mathbf{H}_n$ the nth Wiener chaos of W. We recall that $\mathbf{H}_0$ is simply $\mathbb{R}$ and for $n \geq 1$, $\mathbf{H}_n$ is the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_n(W(h)), h \in \mathcal{H}, \|h\|_{\mathcal{H}} = 1\}$, where H_n is the nth Hermite polynomial. Then we will have the orthogonal decomposition

$$L^2(\Omega) = \oplus_{n=0}^\infty \mathbf{H}_n . \tag{2.6}$$

For each $n \geq 0$, we will denote by J_n the orthogonal projection on the nth Wiener chaos. Consider the one-parameter semigroup $\{T_t, t \geq 0\}$ of contraction operators on $L^2(\Omega)$ defined by

$$T_t(F) = \sum_{n=0}^{\infty} e^{-nt} J_n F\,, \tag{2.7}$$

which is called the Ornstein-Uhlenbeck semigroup. The following property of T_t is taken from [12].

Proposition 2.2 *For any $p > 1$, if $F \in L^p(\Omega)$, then $T_t F \in \mathbb{D}^{1,p}$ for any $t > 0$ and we also have*

$$\lim_{t \to 0} \|T_t F - F\|_{1,p} = 0\,. \tag{2.8}$$

We are ready to give the definition of mild solution to equation (1.1). We denote by $p_t(x)$ the d-dimensional heat kernel $p_t(x) = (2\pi t)^{-d/2} e^{-|x|^2/2t}$, for any $t > 0$, $x \in \mathbb{R}^d$. For each $t \geq 0$, let $\mathcal{F}_t$ be the σ-field generated by the random variables $W(\varphi)$, where φ has support in $[0,t] \times \mathbb{R}^d$. We say that a random field $u(t,x)$ is adapted if for each (t,x) the random variable $u(t,x)$ is $\mathcal{F}_t$-measurable. Then we have the following definition.

Definition 2.3 *An adapted random field $u = \{u(t,x), t \geq 0, x \in \mathbb{R}^d\}$ such that $\mathbb{E}[u^2(t,x)] < \infty$ for all (t,x) is a mild solution to equation (1.1) with bounded initial condition u_0, if for any $(t,x) \in [0,\infty) \times \mathbb{R}^d$, the process $\{p_{t-s}(x-y)u(s,y)\mathbf{1}_{[0,t]}(s), s \geq 0, y \in \mathbb{R}^d\}$ is Skorohod integrable, and the following equation holds*

$$u(t,x) = p_t u_0(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) u(s,y)\, \delta W_{s,y} \quad a.s. \tag{2.9}$$

The following theorem about the existence and uniqueness of the solution to equation (1.1) is taken from [7].

Theorem 2.4 *Suppose that μ satisfies Hypothesis 2.1 and γ is locally integrable. Then equation (1.1) admits a unique mild solution in the sense of Definition 2.3.*

The next lemma states the non-negativity of the solution and will be used in the next section.

Lemma 2.5 *Assume that μ satisfies Hypothesis 2.1 and γ is locally integrable. If the initial condition u_0 is nonnegative, then for each $(t,x) \in [0,\infty) \times \mathbb{R}^d$, $u(t,x) \geq 0$ a.s.*

Proof We will follow the procedure in Section 3.2, [7]. For any $\delta > 0$, we define the function $\varphi_\delta(t) = \frac{1}{\delta}\mathbf{1}_{[0,\delta]}(t)$ for $t \in \mathbb{R}$. Then, $\varphi_\delta(t) p_\varepsilon(x)$ provides an approximation of the Dirac delta function $\delta_0(t,x)$ as ε and δ tend to zero. Define

$$u^{\varepsilon,\delta}(t,x) = \mathbb{E}_B \left[\exp\left(W(A^{\varepsilon,\delta}_{t,x}) - \frac{1}{2}\alpha^{\varepsilon,\delta}_{t,x} \right) \right], \tag{2.10}$$

where

$$A^{\varepsilon,\delta}_{t,x}(r,y) = \frac{1}{\delta}\left(\int_0^{\delta \wedge (t-r)} p_\varepsilon(B^x_{t-r-s} - y) ds \right) \mathbf{1}_{[0,t]}(r), \quad \text{and} \quad \alpha^{\varepsilon,\delta}_{t,x} = \|A^{\varepsilon,\delta}_{t,x}\|^2_{\mathcal{H}}, \tag{2.11}$$

for a standard d-dimensional Brownian motion B independent of W. Then it is obvious from the definition of $u^{\varepsilon,\delta}(t,x)$ that $u^{\varepsilon,\delta}(t,x) > 0$ a.s. for each (t,x), and from Theorem 3.6 in [7] and its proof, we see that for each $F \in \mathbb{D}^{1,2}$ and (t,x), $\mathbb{E}(F u^{\varepsilon,\delta}(t,x))$ converges to $\mathbb{E}(F u(t,x))$. Now for

each fixed (t,x), we take $F = T_s \mathbf{1}_{\{u(t,x)<0\}}$, from Proposition 2.2 we know that such F is in $\mathbb{D}^{1,2}$, for each $s>0$. So we have

$$\mathbb{E}\left(\left(T_s \mathbf{1}_{\{u(t,x)<0\}}\right) u(t,x)\right) = \lim_{\delta,\varepsilon\to 0} \mathbb{E}\left(\left(T_s \mathbf{1}_{\{u(t,x)<0\}}\right) u^{\varepsilon,\delta}(t,x)\right) \geq 0\,,$$

then letting s go to 0 we obtain by Proposition 2.2

$$\mathbb{E}\left(\mathbf{1}_{\{u(t,x)<0\}} u(t,x)\right) \geq 0\,,$$

which shows that $u(t,x) \geq 0$ a.s. ∎

The next result concerning the moment formula for the solution is taken from [7], see also [3], where γ is the Dirac delta function δ.

Theorem 2.6 *Suppose γ is locally integrable and μ satisfies Hypothesis 2.1. Let $u(t,x)$ be the solution to equation (1.1). Then for any integer $p \geq 2$*

$$\mathbb{E}u^p(t,x) = \mathbb{E}_B\left[\prod_{i=1}^p u_0(B_t^i + x) \exp\left(\lambda^2 \sum_{1\leq i<j\leq p} \int_0^t \int_0^t \gamma(s-r)\Lambda(B_s^i - B_r^j) ds dr\right)\right], \qquad (2.12)$$

where $\{B^j,\, j=1,\dots,p\}$ is a family of d-dimensional independent standard Brownian motions independent of W.

3 Main results

We need first to introduce some notation. If μ is a measure satisfying Hypothesis 2.1, for any real number $N>0$, we define

$$C_N = \int_{|\xi|>N} \frac{\mu(d\xi)}{|\xi|^2}, \quad \text{and} \quad D_N = \mu\left\{\xi : |\xi| \leq N\right\}. \qquad (3.1)$$

On the other hand, if γ is locally integrable, we set

$$\int_0^t \gamma(s) ds = \Gamma_t\,. \qquad (3.2)$$

Theorem 3.1 *Let $u(t,x)$ be the solution to equation (1.1) driven by a noise W with covariance structure (2.1). Assume that u_0 is non-negative and supported in the ball $B_M = \{x \in \mathbb{R}^d : |x| \leq M\}$. Assume that μ satisfies Hypothesis 2.1 and γ is locally integrable. Set $\theta_t = \sqrt{D_{N_t} C_{N_t}^{-1}}$, where*

$$N_t = \inf\left\{N \geq 0 : C_N \leq \frac{(2\pi)^d}{32(p-1)\lambda^2 \Gamma_t}\right\}. \qquad (3.3)$$

Then, for any integer $p \geq 2$, we have

$$\bar{\nu}(p) := \inf\left\{\varrho > 0 : \limsup_{t\to\infty} \frac{1}{t\theta_t^2} \sup_{|x|\geq \varrho t\theta_t} \log \mathbb{E}u^p(t,x) < 0\right\} \leq 1\,. \qquad (3.4)$$

Proof Using the moment formula (2.12), together with Cauchy-Schwartz inequality, we can write for any integer $p \geq 2$

$$\mathbb{E}u^p(t,x) = \mathbb{E}_B\left(\prod_{i=1}^p u_0(x + B_t^i) \exp\left(\lambda^2 \sum_{1\leq i<j\leq p} \int_0^t \int_0^t \gamma(s-r)\Lambda(B_s^i - B_r^j) ds dr\right)\right)$$

$$\leq \mathbb{E}_B\left(\prod_{i=1}^p \mathbf{1}_{B_M}(x+B_t^i)\exp\left(\lambda^2\sum_{1\leq i<j\leq p}\int_0^t\int_0^t\gamma(s-r)\Lambda(B_s^i-B_r^j)dsdr\right)\right)\|u_0\|_\infty^p$$

$$\leq \left(\mathbb{E}_B\exp\left(2\lambda^2\sum_{1\leq i<j\leq p}\int_0^t\int_0^t\gamma(s-r)\Lambda(B_s^i-B_r^j)dsdr\right)\right)^{\frac{1}{2}}$$

$$\times\left(\mathbb{E}\mathbf{1}_{B_M}(x+B_t^1)\right)^{\frac{p}{2}}\|u_0\|_\infty^p.$$

Note that in the above expression, the first expectation in the last inequality is exactly the pth moment of the solution to equation (1.1) (denoted by $v(t,x)$) with noise W having a covariance functional with parameters γ and 2Λ respectively, and with initial condition 1. From [7] we see that $v(t,x)$ admits the chaos expansion

$$v(t,x)=\sum_{n=0}^\infty I_n(f_n(\cdot,t,x)),$$

where for $n\geq 1$, the kernel f_n is given by

$$f_n(s_1,x_1,\dots,s_n,x_n,t,x)=\frac{1}{n!}p_{t-s_{\sigma(n)}}(x-x_{\sigma(n)})\cdots p_{s_{\sigma(2)}-s_{\sigma(1)}}(x_{\sigma(2)}-x_{\sigma(1)}).$$

In the above expression, σ is the permutation of $\{1,2,\dots,n\}$ such that $s_{\sigma(1)}<s_{\sigma(2)}<\cdots<s_{\sigma(n)}<s_{\sigma(n+1)}=t$. We have

$$\mathbb{E}[I_n(f_n(\cdot,t,x))^2]=n!\|f_n(\cdot,t,x)\|^2_{\mathcal{H}_1^{\otimes n}},$$

where $\mathcal{H}_1$ denotes the norm introduced in (2.2), but with Λ replaced by 2Λ. By the hypercontractivity property, we have

$$\|I_n(f_n(\cdot,t,x))\|_{L^p(\Omega)}\leq (p-1)^{\frac{n}{2}}\|I_n(f_n(\cdot,t,x))\|_{L^2(\Omega)}.$$

Therefore, have the L^p norm of the $v(t,x)$ is bounded as follows

$$\begin{aligned}\|v(t,x)\|_{L^p(\Omega)} &\leq \sum_{n=0}^\infty\|I_n(f_n(\cdot,t,x))\|_{L^p(\Omega)}\leq\sum_{n=0}^\infty(p-1)^{\frac{n}{2}}\|I_n(f_n(\cdot,t,x))\|_{L^2(\Omega)}\\ &= \sum_{n=0}^\infty(p-1)^{\frac{n}{2}}\sqrt{n!}\|f_n(\cdot,t,x)\|_{\mathcal{H}_1^{\otimes n}}.\end{aligned}$$

Then using Fourier transform as in [7] we have

$$\begin{aligned}n!\|f_n(\cdot,t,x)\|^2_{\mathcal{H}_1^{\otimes n}} &\leq \frac{(2\lambda^2)^n n!}{(2\pi)^{nd}}\int_{[0,t]^{2n}}\int_{\mathbb{R}^{nd}}|\mathcal{F}f_n(s,\cdot,t,x)(\xi)|^2\mu(d\xi)\prod_{i=1}^n\gamma(s_i-r_i)dsdr\\ &\leq \frac{(4\lambda^2\Gamma_t)^n n!}{(2\pi)^{nd}}\int_{[0,t]^n}\int_{\mathbb{R}^{nd}}|\mathcal{F}f_n(s,\cdot,t,x)(\xi)|^2\mu(d\xi)ds\\ &\leq \frac{(4\lambda^2\Gamma_t)^n}{(2\pi)^{nd}}\int_{\mathbb{R}^{nd}}\int_{T_n(t)}\prod_{i=1}^n e^{-(s_{i+1}-s_i)|\xi_i|^2}ds\mu(d\xi),\end{aligned}$$

where $T_n(t)$ denotes the simplex $\{0\leq s_1\leq s_2\leq\cdots\leq s_n\leq t\}$, $\mu(d\xi)=\prod_{i=1}^n\mu(d\xi_i)$ and ds is defined similarly. Then with the change of variable $s_{i+1}-s_i=w_i$ and by Lemma 3.3 below applied to $N=N_t$, we obtain

$$n!\|f_n(\cdot,t,x)\|^2_{\mathcal{H}_1^{\otimes n}}\leq\frac{(4\lambda^2\Gamma_t)^n}{(2\pi)^{nd}}\int_{S_{t,n}}\int_{\mathbb{R}^{nd}}\prod_{i=1}^n e^{-w_i|\xi_i|^2}\mu(d\xi)dw$$

$$
\begin{aligned}
&\leq \frac{(4\lambda^2\Gamma_t)^n}{(2\pi)^{nd}} \sum_{k=0}^n \binom{n}{k} \frac{t^k}{k!} D_{N_t}^k C_{N_t}^{n-k} \\
&\leq \left(\frac{8\lambda^2\Gamma_t C_{N_t}}{(2\pi)^d}\right)^n \sum_{k=0}^\infty \frac{t^k D_{N_t}^k C_{N_t}^{-k}}{k!} \\
&= \left(\frac{8\lambda^2\Gamma_t C_{N_t}}{(2\pi)^d}\right)^n e^{tD_{N_t} C_{N_t}^{-1}} ,
\end{aligned}
$$

where $S_{t,n} = \{(w_1, \dots, w_n) \in [0,\infty)^n : w_1 + \dots + w_n \leq t\}$. Thus

$$
\|v(t,x)\|_{L^p(\Omega)} \leq \sum_{n=0}^\infty \left(\frac{8\lambda^2\Gamma_t C_{N_t}}{(2\pi)^d}\right)^{\frac{n}{2}} (p-1)^{\frac{n}{2}} e^{\frac{1}{2}tD_{N_t} C_{N_t}^{-1}} \leq 2e^{\frac{1}{2}tD_{N_t} C_{N_t}^{-1}} ,
$$

where the last inequality comes from the definition of N_t. Thus we obtain from Lemma 3.5 below

$$
\mathbb{E}u(t,x)^p \leq 2^{\frac{p}{2}} e^{\frac{p}{4}tD_{N_t} C_{N_t}^{-1}} \frac{1}{(2\pi t)^{\frac{dp}{4}}} e^{-\frac{|x|^2 p}{4t(\varkappa+1)}} e^{\frac{M^2 p}{4t\varkappa}} \omega_d^{\frac{p}{2}} M^{\frac{dp}{2}} \|u_0\|_\infty^p
$$

for any $\varkappa > 0$. As a consequence, if we want

$$
\limsup_{t\to\infty} \frac{1}{t\theta_t^2} \log \sup_{|x|\geq \varrho t\theta_t} \mathbb{E}u(t,x)^p \leq \frac{p}{4} - \frac{p\varrho^2}{4(\varkappa+1)} < 0 ,
$$

we need $\varrho > \sqrt{\varkappa+1}$. Letting $\varkappa \to 0$ we conclude that $\bar{\nu}(p) \leq 1$. ■

Section 6 in [7] gives the moment upper bounds for some specific choices of γ and Λ, assuming the initial condition is a bounded function. Actually the proof of Theorem 3.1 above also gives a general upper bound for the pth moment, stated in the following corollary.

Corollary 3.2 *Let $u(t,x)$ be the solution to equation (1.1) with a bounded nonnegative initial condition. Let D_N, C_N be defined as in (3.1) and N_t be defined as in (3.3). Then we have the moment upper bound*

$$
\mathbb{E}u^p(t,x) \leq C^p \exp\left(CptD_{N_t} C_{N_t}^{-1}\right) , \tag{3.5}
$$

for some constant C independent of p and t.

The next lemma is used in the proof of Theorem 3.1. For a proof, see Lemma 3.3 in [7].

Lemma 3.3 *Let μ satisfy Hypothesis 2.1. For any $N > 0$ let D_N and C_N be given by (3.1). Then we have*

$$
\int_{\mathbb{R}^{nd}} \int_{S_{t,n}} e^{-\sum_{i=1}^n w_i |\xi_i|^2} dw \mu(d\xi) \leq \sum_{k=0}^n \binom{n}{k} \frac{t^k}{k!} D_N^k C_N^{n-k} .
$$

The next result is a lower bound for the lower intermittency front, when Λ is bounded below by the Riesz kernel.

Theorem 3.4 *Let $u(t,x)$ be the solution to equation (1.1) with nonnegative initial condition u_0 being uniformly bounded away from 0 in the ball B_M. Assume that*

$$
\Lambda(x) \geq C_\Lambda |x|^{-\beta}, \ \forall |x| \leq R, \ \text{for some } R > 0 \tag{3.6}
$$

with $0 \leq \beta < 2 \wedge d$. Suppose that

$$
\lim_{t\to\infty} \Gamma_t = \infty. \tag{3.7}
$$

Fix $\delta \in (0,1)$ and set $\eta_t = \Gamma_{t\delta^2}^{\frac{1}{2-\beta}}$. Define

$$\underline{\nu}(p) := \sup\left\{\varrho > 0 : \limsup_{t\to\infty} \frac{1}{t\eta_t^2} \sup_{|x|\geq \varrho t\eta_t} \log E(|u(t,x)|^p) > 0\right\}. \tag{3.8}$$

Then we have

$$\underline{\nu}(p) \geq \sqrt{C_{\beta,\delta}}\lambda^{\frac{2}{2-\beta}}(p-1)^{\frac{1}{2-\beta}}, \tag{3.9}$$

where

$$C_{\beta,\delta} = 2\left[\left(\frac{\beta}{2}\right)^{\frac{\beta}{2-\beta}} - \left(\frac{\beta}{2}\right)^{\frac{2}{2-\beta}}\right](1-\delta)^{\frac{2}{2-\beta}}\delta j_\nu^{\frac{-2\beta}{2-\beta}} C_\Lambda^{\frac{2}{2-\beta}}\sqrt{2(1-\delta)},$$

and j_ν denotes the smallest positive zero of the Bessel function $J_\nu(x)$ of index $\nu = \frac{d-2}{2}$.

Proof Note that using the change of variable $s \to \frac{u+v}{2}$ and $r \to \frac{v-u}{2}$ we have

$$\begin{aligned}\int_0^t\int_0^t \gamma(s-r)dsdr &= \int_{-t}^t\int_u^{2t-u}\gamma(u)dvdu = 4\int_0^t \gamma(u)(t-u)du \\ &\geq 4(1-\delta)t\int_0^{t\delta}\gamma(u)du = 4(1-\delta)t\Gamma_{t\delta}.\end{aligned}$$

Let $u_0(x) \geq C_{u_0}\mathbf{1}_M(x)$. Then using the moment formula for the solution $u(t,x)$ as before,

$$\begin{aligned}\mathbb{E}u^p(t,x) &= \mathbb{E}_B\left(\prod_{i=1}^p u_0(x+B_t^i)\exp\left(\lambda^2\sum_{1\leq i<j\leq p}\int_0^t\int_0^t\gamma(s-r)\Lambda(B_s^i - B_r^j)dsdr\right)\right) \\ &\geq \mathbb{E}_B\left(\prod_{i=1}^p u_0(x+B_t^i)\exp\left(\lambda^2\sum_{1\leq i<j\leq p}\int_0^{t\delta}\int_0^{t\delta}\gamma(s-r)\Lambda(B_s^i - B_r^j)dsdr\right)\right) \\ &\geq C_{u_0}^p P\left(\sup_{1\leq i\leq p}|B_t^i + x| \leq M, \sup_{0\leq s\leq t\delta, 1\leq i\leq p}|B_s^i| \leq \varepsilon\right) \\ &\quad\times\exp\left(2\lambda^2 p(p-1)(1-\delta)\delta t\Gamma_{t\delta^2}C_\Lambda|2\varepsilon|^{-\beta}\right) \\ &= C_{u_0}^p P\left(|B_t^0 + x| \leq M, \sup_{0\leq s\leq t\delta}|B_s^0| \leq \varepsilon\right)^p \\ &\quad\times\exp\left(2\lambda^2 p(p-1)(1-\delta)\delta t\Gamma_{t\delta^2}C_\Lambda|2\varepsilon|^{-\beta}\right),\end{aligned}$$

where B_s^0 is a standard Brownian motion, ε is a positive number satisfying $\varepsilon < \frac{R}{2}$, which will be chosen later. In order to estimate of the above probability, notice that

$$\begin{aligned}&P\left(|B_t^0 + x| \leq M, \sup_{0\leq s\leq t\delta}|B_s^0| \leq \varepsilon\right) \\ &= \mathbb{E}\left(\mathbb{E}\left\{\mathbf{1}_{\{|B_t^0 - B_{t\delta}^0 + x + B_{t\delta}^0| \leq M, \sup_{0\leq s\leq t\delta}|B_s^0|\leq\varepsilon\}}\Big|\mathcal{G}_{t\delta}\right\}\right),\end{aligned}$$

where $\mathcal{G}_t$ is the filtration generated by $\{B_s^0 : 0 \leq s \leq t\}$. Then we can choose $\varepsilon \leq \frac{M}{2}$(the specific choice of ε will be given below) and invoke Lemma 3.5 to get, for t large enough,

$$P\left(|B_t^0 + x| \leq M, \sup_{0\leq s\leq t\delta}|B_s^0| \leq \varepsilon\right) \geq \mathbb{E}\left(\mathbb{E}\left\{\mathbf{1}_{\{|B_t^0 - B_{t\delta}^0 + x| \leq \frac{M}{2}}\Big|\mathcal{G}_{t\delta}\right\}\mathbf{1}_{\{\sup_{0\leq s\leq t\delta}|B_s^0|\leq\varepsilon\}}\right)$$

$$
\begin{aligned}
&\geq P\left\{|B^0_{t(1-\delta)}+x|\leq \frac{M}{2}\right\} P\left\{\sup_{0\leq s\leq 1}|B^0_s|\leq \frac{\varepsilon}{\sqrt{t\delta}}\right\}\\
&\geq C\frac{\omega_d M^d}{(t(1-\delta))^{\frac{d}{2}}} e^{-\frac{M^2(1+\frac{1}{\varkappa})}{8t(1-\delta)}} e^{-\frac{(\varkappa+1)|x|^2}{2t(1-\delta)}} e^{-\frac{j_\nu^2 t\delta}{2\varepsilon^2}},
\end{aligned}
$$

where C is a universal constant. The last inequality follows from the small ball probability estimate (see Theorem 1 in [13])

$$
P\left\{\sup_{0\leq s\leq 1}|B_s|\leq \varepsilon\right\}\sim e^{-\frac{j_\nu^2}{2\varepsilon^2}}, \quad \text{as} \quad \varepsilon\to 0\,. \tag{3.10}
$$

Then we obtain

$$
\begin{aligned}
\mathbb{E}u^p(t,x) \quad \geq \quad & (CC_{u_0})^p (t(1-\delta))^{-\frac{dp}{2}} \exp\Bigg(-\frac{M^2(1+\frac{1}{\varkappa})}{8t(1-\delta)} - \frac{p(\varkappa+1)|x|^2}{2t(1-\delta)} - \frac{pj_\nu^2 t\delta}{2\varepsilon^2}\\
& +2\lambda^2 p(p-1)(1-\delta)\delta t\Gamma_{t\delta^2} C_\Lambda |2\varepsilon|^{-\beta}\Bigg).
\end{aligned}
$$

We apply Lemma 3.6 with

$$
A = 2^{-\beta+1}\lambda^2 p(p-1)(1-\delta)\delta t\Gamma_{t\delta^2}C_\Lambda \quad \text{and} \quad B=\frac{j_\nu^2 t\delta p}{2}
$$

to maximize the right hand side of the above inequality, by choosing

$$
\varepsilon = \left(\frac{2^{\beta-1}j_\nu^2}{\beta\lambda^2(p-1)(1-\delta)\Gamma_{t\delta^2}C_\Lambda}\right)^{\frac{1}{2-\beta}}.
$$

In this way we obtain

$$
\mathbb{E}u^p(t,x)\geq (CC_{u_0})^p(t(1-\delta))^{-\frac{dp}{2}}\exp\left(-\frac{M^2(1+\frac{1}{\varkappa})}{8t(1-\delta)} - \frac{p(\varkappa+1)|x|^2}{2t(1-\delta)} + C_{\beta,\delta}\lambda^{\frac{4}{2-\beta}}p(p-1)^{\frac{2}{2-\beta}}t\Gamma_{t\delta^2}^{\frac{2}{2-\beta}}\right).
$$

We remark condition (3.7) implies that ε chosen above tends to 0 as $t\to\infty$, thus the small ball estimate used above works for t large enough, in such a way that $\varepsilon<\frac{M}{2}$. If we want

$$
\limsup_{t\to\infty}\frac{1}{t\eta_t^2}\log\sup_{|x|\geq\varrho t\eta_t}\mathbb{E}u^p(t,x)\geq \frac{-p(\varkappa+1)\varrho^2}{2(1-\delta)} + C_{\beta,\delta}\lambda^{\frac{4}{2-\beta}}p^{\frac{4-\beta}{2-\beta}}>0\,.
$$

we need

$$
\varrho<\frac{\sqrt{C_{\beta,\Lambda}}}{\sqrt{\varkappa+1}}\lambda^{\frac{2}{2-\beta}}(p-1)^{\frac{1}{2-\beta}},
$$

Letting $\varkappa\to 0$ and invoking Lemma 2.5 we conclude that

$$
\underline{\nu}(p)\geq\sqrt{C_{\beta,\delta}}\lambda^{\frac{2}{2-\beta}}(p-1)^{\frac{1}{2-\beta}}.
$$

The theorem is proved. ■

The next two lemmas are used in the proof of previous theorems.

Lemma 3.5 *For any positive M and $\varkappa$ we have*

$$
\frac{1}{(2\pi t)^{\frac{d}{2}}}e^{-\frac{(\varkappa+1)|x|^2}{2t}}e^{-\frac{M^2(1+\frac{1}{\varkappa})}{2t}}\omega_d M^d \leq \int_{|y|\leq M}\frac{1}{(2\pi t)^{\frac{d}{2}}}e^{-\frac{|y-x|^2}{2t}}dy \leq \frac{1}{(2\pi t)^{\frac{d}{2}}}e^{-\frac{|x|^2}{2t(\varkappa+1)}}e^{\frac{M^2}{2t\varkappa}}\omega_d M^d, \tag{3.11}
$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$.

Proof We have the simple fact that

$$|y-x|^2 \le |y|^2 + 2|x||y| + |x|^2 \le |y|^2 + \frac{|y|^2}{\varkappa} + \varkappa|x|^2 + |x|^2 = (1+\frac{1}{\varkappa})|y|^2 + (\varkappa+1)|x|^2 \,,$$

and from this we deduce the reverse inequality

$$|x-y|^2 \ge \frac{|x|^2}{\varkappa+1} - \frac{|y|^2}{\varkappa} \,,$$

from these two inequalities the proof is done by elementary calculations. ■

Lemma 3.6 *Let $A, B > 0$ and $0 < \beta < 2$. Then the function*

$$f(x) = Ax^{-\beta} - Bx^{-2}$$

attains its maximum at $x = \left(\frac{2B}{\beta A}\right)^{\frac{1}{2-\beta}}$, and the maximum value equals

$$\left(\left(\frac{\beta}{2}\right)^{\frac{\beta}{2-\beta}} - \left(\frac{\beta}{2}\right)^{\frac{2}{2-\beta}}\right) A^{\frac{2}{2-\beta}} B^{\frac{-\beta}{2-\beta}}.$$

Remark 3.7 *If condition (3.7) does not hold, that is, the limit is finite Γ_∞ (which happens, for instance, if γ is a Dirac function), then, we need the following additional condition on M:*

$$R \wedge M \ge 2\left(\frac{2^{\beta-1} j_\nu^2}{\beta\lambda^2(p-1)(1-\delta)\Gamma_\infty C_\Lambda}\right)^{\frac{1}{2-\beta}} . \tag{3.12}$$

Remark 3.8 *When γ is the Dirac delta function, the noise $\dot{W}$ is white in time and correlated in space and Theorems 3.1 and 3.4 still hold with $\Gamma_t = \frac{1}{2}$. The appearance of the functions θ_t and η_t in Theorems 3.1 and 3.4 come from the time covariance of the noise. In the case when $\Gamma_t \to \infty$ as $t \to \infty$ (for instance, when $\gamma(t) = |t|^{-\alpha}$ with $0 < \alpha < 1$, $\Gamma_t = \frac{t^{1-\alpha}}{1-\alpha}$), the restriction on M (3.12) is not needed.*

In the case where $\Lambda(x) = |x|^{-\beta}$ with $0 < \beta < 2 \wedge d$, we obtain the following more precise result concerning the upper bound of the intermittency front. In this case, the function θ_t defined in Theorem 3.1 can be replaced by $\Gamma_t^{\frac{1}{2-\beta}}$. Notice that this function coincides with the function η_t in Theorem 3.4 except for the factor δ^2. If we let δ tend to 1, then the lower bound in (3.8) tends to zero.

Proposition 3.9 *Assume that u_0 is non-negative and supported in the ball B_M. Let γ be a locally integrable, positive and positive definite function, and $\Lambda(x) = |x|^{-\beta}$, assume $0 < \beta < 2 \wedge d$. Set $\vartheta_t = \Gamma_t^{\frac{1}{2-\beta}}$, where Γ_t is defined in (3.2). Define*

$$\bar{\upsilon}(p) := \inf\left\{\varrho > 0 : \limsup_{t\to\infty} \frac{1}{t\vartheta_t^2} \sup_{|x|\ge\varrho t\vartheta_t} \log \mathbb{E}u^p(t,x) < 0\right\} , \tag{3.13}$$

then

$$\bar{\upsilon}(p) \le 2\sqrt{2}\left(\frac{\Gamma\left(\frac{d-\beta}{2}\right)\Gamma\left(1-\frac{\beta}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\right)^{\frac{1}{2-\beta}} (p-1)^{\frac{1}{2-\beta}} \lambda^{\frac{2}{2-\beta}} .$$

Proof We will follow the notations and the same calculations used in the proof of Theorem 3.1. We have

$$n!\|f_n(\cdot,t,x)\|^2_{\mathcal{H}_1^{\otimes n}} \le \frac{(4\lambda^2\Gamma_t)^n}{(2\pi)^{nd}}\int_{\mathbb{R}^{nd}}\int_{T_n(t)}\prod_{i=1}^n e^{-(s_{i+1}-s_i)|\xi_i|^2}ds\mu(d\xi)\,.$$

Since $\Lambda(x)=|x|^{-\beta}$, its Fourier transform is given by (see, e.g., Chapter 5 in [14])

$$\mu(d\xi)=\frac{\pi^{\frac{d}{2}}2^{d-\beta}\Gamma\left(\frac{d-\beta}{2}\right)}{\Gamma(\frac{\beta}{2})}|\xi|^{\beta-d}d\xi := \Lambda_\beta|\xi|^{\beta-d}d\xi\,.$$

Using polar coordinates we can compute the integral

$$\int_{\mathbb{R}^d}e^{-|\eta|^2}|\eta|^{\beta-d}d\eta=\frac{\pi^{\frac{d}{2}}\Gamma(\frac{\beta}{2})}{\Gamma(\frac{d}{2})}\,.$$

Then with the change of variable $\sqrt{s_{i+1}-s_i}\xi_i\to\eta_i$, we have

$$\begin{aligned}
n!\|f_n(\cdot,t,x)\|^2_{\mathcal{H}_1^{\otimes n}} &\le \frac{(4\lambda^2\Gamma_t\Lambda_\beta)^n}{(2\pi)^{nd}}\int_{T_n(t)}\int_{\mathbb{R}^{nd}}\prod_{i=1}^n\left(e^{-|\eta_i|^2}|\eta_i|^{\beta-d}\right)\prod_{i=1}^n(s_{i+1}-s_i)^{-\frac{\beta}{2}}d\eta ds\\
&= \frac{(4\lambda^2\Gamma_t\Lambda_\beta\pi^{\frac{d}{2}})^n}{(2\pi)^{nd}\Gamma(\frac{d}{2})^n}\Gamma(\frac{\beta}{2})^n\int_{T_n(t)}\prod_{i=1}^n(s_{i+1}-s_i)^{-\frac{\beta}{2}}ds\\
&= \frac{(4\lambda^2\Gamma_t\Lambda_\beta\pi^{\frac{d}{2}})^n}{(2\pi)^{nd}\Gamma(\frac{d}{2})^n}\Gamma(\frac{\beta}{2})^n\frac{\Gamma(1-\frac{\beta}{2})^n t^{n(1-\frac{\beta}{2})}}{\Gamma((1-\frac{\beta}{2})n+1)}\,.
\end{aligned}$$

To alleviate the notation we denote

$$B:=\frac{\Lambda_\beta\Gamma(\frac{\beta}{2})\Gamma(1-\frac{\beta}{2})}{2^{d-2}\pi^{\frac{d}{2}}\Gamma(\frac{d}{2})}\,.$$

By the log-convexity of Gamma function, we know that $\Gamma(1+x)^2\le\Gamma(1+2x)$. Then we obtain

$$\begin{aligned}
\|v(t,x)\|_{L^p(\Omega)} &\le \sum_{n=0}^\infty(B(p-1)\lambda^2\Gamma_t)^{\frac{n}{2}}\frac{t^{n\frac{2-\beta}{4}}}{\Gamma((1-\frac{\beta}{2})n+1)^{\frac{1}{2}}}\\
&\le \sum_{n=0}^\infty(B(p-1)\lambda^2\Gamma_t)^{\frac{n}{2}}\frac{t^{n(\frac{1}{2}-\frac{\beta}{4})}}{\Gamma(n\frac{2-\beta}{4}+1)}\,.
\end{aligned}$$

Using the asymptotic behavior of Mittag-Leffler function (see e.g., page 208 in [6])

$$\sum_{n=0}^\infty\frac{z^n}{\Gamma(1+an)}=\frac{1}{a}\exp(z^{\frac{1}{a}})+O(|z|^{-1})\quad\text{as } z\to\infty\,,$$

and considering the fact that we are only interested in when $t\to\infty$, we obtain

$$\|v(t,x)\|_{L^p(\Omega)}\le\frac{4}{2-\beta}\exp\left((B(p-1)\lambda^2\Gamma_t)^{\frac{1}{1-\frac{\beta}{2}}}t\right)+O(t^{-\frac{2-\beta}{4}})\,.$$

Thus, as in the proof of Theorem 3.1, we obtain

$$\mathbb{E}u^p(t,x)\le\left(\frac{4}{2-\beta}\right)^{\frac{p}{2}}\exp\left(\frac{p}{2}(B(p-1)\lambda^2\Gamma_t)^{\frac{2}{2-\beta}}t-\frac{p|x|^2}{4t(\varkappa+1)}\right)e^{\frac{pM^2}{4t\varkappa}}\omega_d^{\frac{p}{2}}M^{\frac{dp}{2}}\frac{1}{(2\pi t)^{\frac{dp}{4}}}\,.$$

Recall that we have set $\vartheta_t = \Gamma_t^{\frac{1}{2-\beta}}$. Thus, if we want

$$\limsup_{t\to\infty} \frac{1}{t\vartheta_t^2} \sup_{|x|\geq \varrho t\vartheta_t} \log \mathbb{E} u^p(t,x) \leq p\left(B(p-1)\lambda^2\right)^{\frac{1}{1-\frac{\beta}{2}}} - \frac{p\varrho^2}{2(\varkappa+1)} < 0\,,$$

by invoking Lemma 2.5 and letting $\varkappa \to 0$, we conclude that

$$\bar{\upsilon}(p) \leq \sqrt{2}\left(B(p-1)\lambda^2\right)^{\frac{1}{2-\beta}}\,.$$

Finally, if we plug in the value of B and Λ_β, the proposition is proved.

■

Remark 3.10 *Proposition 3.9 still holds if we take the fractional kernel $\Lambda(x) = \prod_{i=1}^d |x_i|^{2H_i-2}$ with $\frac{1}{2} < H_i < 1$ for all i and $\beta := 2d - 2\sum_{i=1}^d H_i$ with $0 < \beta < 2$. The order of $p-1$ and λ in the upper bound of $\bar{\upsilon}(p)$ will be exactly the same, although the coefficient may be different.*

Theorem 3.4 does not cover the case when the noise is white in space. However, if we approximate the Dirac delta function by $p_\varepsilon(x)$, we have the following result.

Proposition 3.11 *Assume $d = 1$. Let $u(t,x)$ be the solution to equation (1.1) with nonnegative initial condition u_0 being uniformly bounded away from 0 in the ball B_M and supported in B_{rM}, where $r \geq 1$. Assume that $\Lambda(x)$ is the Dirac delta function. Set $\vartheta_t = \Gamma_t$, fix $\delta \in (0,1)$ and set $\eta_t = \Gamma_{t\delta^2}$. Let $\bar{\upsilon}(p)$ and $\underline{\upsilon}(p)$ be defined in (3.13) and (3.8), respectively. Then we have*

$$\bar{\upsilon}(p) \leq 2\sqrt{2}(p-1)\lambda^2\,. \tag{3.14}$$

If we further assume (3.7) holds, then

$$\underline{\upsilon}(p) \geq \frac{2\sqrt{2}}{e^2\pi^{\frac{3}{2}}}(1-\delta)^{\frac{3}{2}}\delta^{\frac{1}{2}}(p-1)\lambda^2\,. \tag{3.15}$$

Proof The proof of upper bound follows along the same lines as the proof for proposition 3.9 except now $\mu(d\xi) = d\xi$. For the lower bound, we consider the approximation of the Dirac delta function by the heat kernel p_ε, and define

$$I_{t,p,\epsilon} = \mathbb{E}_B\left(\prod_{i=1}^p u_0(x+B_t^i)\exp\left(\lambda^2 \sum_{1\leq i<j\leq p} \int_0^t\int_0^t \gamma(s-r)p_\varepsilon(B_s^i - B_r^j)dsdr\right)\right)\,.$$

Expanding the exponential and using Fourier analysis, one can show that $\mathbb{E}u(t,x)^p \geq I_{t,p,\varepsilon}$(see [7, 8]) for any $\varepsilon > 0$. Then the proof follows along the same lines as the proof for Theorem 3.4 except here, we restrict the expectation on the set

$$F := \left\{\omega : \sup_{1\leq i\leq p} |B_t^i + x| \leq M,\ \sup_{1\leq i\leq p}\sup_{0\leq s\leq t\delta} |B_s^i| \leq \sqrt{\varepsilon}\right\}\,.$$

We omit the details of the proof. ■

Remark 3.12 *If condition (3.7) does not hold and the limit is finite Γ_∞, then we need the following additional condition on M:*

$$M \geq \frac{\sqrt{2}\pi^{\frac{5}{2}}e^2}{4\lambda^2(p-1)(1-\delta)\Gamma_\infty}\,.$$

References

[1] L. Chen, R. Dalang: Moments and growth indices for the nonlinear stochastic heat equation with rough initial conditions. *Ann. Probab*, to appear.

[2] L. Chen, R. Dalang: Moments, Intermittency and Growth Indices for the Nonlinear Fractional Stochastic Heat Equation. *Stoch. Partial Differ. Equ. Anal. Comput.*, to appear.

[3] D. Conus: Moments for the parabolic Anderson model: on a result by Hu and Nualart. *Commun. Stoch. Anal.* 7 (2013), no. 1, 125-152.

[4] D. Conus, D. Khoshnevisan: On the existence and position of the farthest peaks of a family of stochastic heat and wave equations. *Probab. Theory Related Fields.* 152 (2012), no. 3-4, 681-701.

[5] R. Dalang: Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.'s. *Electron. J. Probab.* 4 (1999), no. 6, 29 pp.

[6] A. Erdélyi, W. Magnus, F. Oberhettinger, F. Tricomi: Higher transcendental functions. Vol. III. Based on notes left by Harry Bateman. Reprint of the 1955 original. *Robert E. Krieger Publishing Co., Inc., Melbourne, Fla.*, 1981. xvii+292 pp.

[7] Y. Hu, J. Huang, D. Nualart, S. Tindel: Stochastic heat equations with general multiplicative Gaussian noises: Hölder continuity and intermittency. *Electron. J. Probab* 20 (2015), no. 55, 50 pp.

[8] Y. Hu, D. Nualart: Stochastic heat equation driven by fractional noise and local time. *Probab. Theory Related Fields* 143 (2009), no. 1-2, 285-328.

[9] D. Khoshnevisan: Analysis of stochastic partial differential equations. CBMS Regional Conference Series in Mathematics, 119. *Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI,* 2014. viii+116 pp.

[10] W. Li, Q. Shao: Gaussian processes: inequalities, small ball probabilities and applications. *Stochastic processes: theory and methods*, 533-597, Handbook of Statist., 19, North-Holland, Amsterdam, 2001.

[11] D. Nualart: The Malliavin Calculus and Related Topics. Second edition. Probability and its Applications (New York). *Springer-Verlag*, Berlin, 2006.

[12] D. Nualart: Malliavin calculus and its applications. CBMS Regional Conference Series in Mathematics, 110. *Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI*, 2009. viii+85 pp.

[13] Z. Shi: Small ball probabilities for a Wiener process under weighted sup-norms, with an application to the supremum of Bessel local times. *J. Theoret. Probab.* 9 (1996), no. 4, 915-929.

[14] E. Stein: Singular integrals and differentiability properties of functions. Princeton Mathematical Series, No. 30 *Princeton University Press, Princeton, N.J.* 1970 xiv+290 pp.